Varianta 066

Subiectul I

a) $\sqrt{7}$. b) $2\sqrt{2}$ .c) $C(c,2):(\mathrm{x}-1)^2+(y-2)^2=4$. d) $S_{ABC}=1$.

e) $\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 1 \end{vmatrix}=0 \Leftrightarrow x+y+z-6=0$ .f) $\mathrm{a}=-3, \mathrm{b}=2\sqrt{10}$.

Subiectul II

1. a) grad P = 0 $\Rightarrow 3$ polinoame. grad P = 1 $\Rightarrow 6$ polinoame. grad P = 2 $\Rightarrow 18$ polinoame. deci 27 de polinoame de grad mai mic sau egal cu 2 in $\mathbf{Z}_3[x]$.

b)Probabilitatea cerută este $\frac{4}{5}$ .c) numarul termenilor raționali este 11. d) $\log_2 3>\log_3 2$.

e) Dacă x_1, x_2, x_3, sunt radacinile polinomului $f=x^3+x^2+1$, atunci

$x_1^2+x_2^2+x_3^2=(x_1+x_2+x_3)^2-2\cdot(x_1x_2+x_1x_3+x_2x_3)=(-1)^2-2\cdot 0=1$.

2. $f:\mathbf{R}\to\mathbf{R}$, $\mathrm{f}(\mathrm{x})=2^x+x^2$

a) $f'(x)=2^x\ln 2+2x$. b) $\int_0^1 f(x)dx=\frac{1}{\ln 2}+\frac{1}{3}$.

c) $f''(x)=2^x\cdot\ln^2 2+2>0, \forall \mathrm{x}\in R\Rightarrow$ f este convexă pe **R**.

d) $\lim\limits_{x\to 1}\frac{f(x)-f(1)}{x-1}=2(\ln 2+1)$. e) $\lim\limits_{x\to 1}\frac{f'(x)}{f(x)}=\lim\limits_{x\to 1}\frac{2^x\cdot\ln 2+2x}{2^x+x^2}=\frac{2\ln 2+2}{3}$.

Subiectul III

a) det A = $\begin{vmatrix} 0 & -3 \\ 1 & 2 \end{vmatrix}=3\neq 0\Rightarrow$ rang A=2. b) $f(O_2)=A\cdot O_2+O_2\cdot A=O_2$,

$f(I_2)=A\cdot I_2+I_2\cdot A=A+A=2A$.

c) $f(aX)=A\cdot aX+aX\cdot A=a\cdot AX+a\cdot XA=a\cdot(AX+XA)=a\cdot f(X)\Rightarrow$

$\Rightarrow f(aX)=af(x), \forall \mathrm{x}\in \mathrm{M}_2(\mathbf{R}), \forall \mathrm{a}\in\mathbf{R}$.

d) $f(X+Y)=A(X+Y)+(X+Y)A=A\cdot X+A\cdot Y+X\cdot A+Y\cdot A=$

$=(AX+XA)+(AY+YA)=f(X)+f(Y)\Rightarrow f(X+Y)=f(X)+f(Y), \forall \mathrm{X},\mathrm{Y}\in\mathrm{M}_2(\mathbf{R})$.

e) Se arata ca in spatiul vectorial $M_2(\mathbf{R})$ multimea $B=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}$

formeaza o baza.

f) si g) Daca $X=\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$ si $Y=\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ din $f(X)=Y$ obtinem sistemul de ecuatii liniare: $\begin{cases} x_2-3x_3 = y_1 \\ -3x_1+2x_3-3x_4 = y_2 \\ x_1+2x_2+x_4 = y_3 \\ x_2-3x_3+4x_4 = y_4 \end{cases}$ cu determinantul $\Delta=\begin{vmatrix} 0 & 1 & -3 & 0 \\ -3 & 0 & 2 & -3 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -3 & 4 \end{vmatrix}\neq 0$.

Sistemul are solutie unica iar pentru Y = 0 obtinem X = 0.

Subiectul IV

a) $1+a+a^2+...+a^n+\dfrac{a^{n+1}}{1-a}=\dfrac{a^{n+1}-1}{a-1}+\dfrac{a^{n+1}}{1-a}=\dfrac{a^{n+1}-1}{a-1}-\dfrac{a^{n+1}}{a-1}=\dfrac{-1}{a-1}=\dfrac{1}{1-a}$, $\forall n\in\mathbf{N}$ $\forall a\in\mathbf{R}\text{-}\{1\}$.

b) Se inlocuieste a in relatia de la punctual a) cu $-\sqrt[3]{x}$.

c) Avem $\dfrac{\left(\sqrt[3]{x}\right)^{n+1}}{1+\sqrt[3]{x}}\geq 0, \forall x\in[0,1], \forall n\in\mathbf{N}^*$ iar

$\dfrac{\left(\sqrt[3]{x}\right)^{n+1}}{1+\sqrt[3]{x}}\leq\left(\sqrt[3]{x}\right)^{n+1}\Leftrightarrow\left(\sqrt[3]{x}\right)^{n+1}\leq\left(\sqrt[3]{x}\right)^{n+1}+\left(\sqrt[3]{x}\right)^{n+2}\Leftrightarrow 0\leq\left(\sqrt[3]{x}\right)^{n+2}$ $\forall x\in[0,1], \forall n\in\mathbf{N}^*$. Deci

$0\leq\dfrac{\left(\sqrt[3]{x}\right)^{n+1}}{1+\sqrt[3]{x}}\leq\left(\sqrt[3]{x}\right)^{n+1}, \forall x\in[0,1], \forall n\in\mathbf{N}^*$.

d) Se integrează relațiile de la punctul c) şi rezulta $0\leq\int_0^b\dfrac{\left(\sqrt[3]{x}\right)^{n+1}}{1+\sqrt[3]{x}}dx\leq\dfrac{b^{\frac{n+1}{3}+1}}{\frac{n+1}{3}+1}$ şi trecând

la limită, având in vedere faptul că $\lim\limits_{n\to\infty}\dfrac{b^{\frac{n+1}{3}+1}}{\frac{n+1}{3}+1}=0$, unde $b\in[0,1]$, obținem

$\lim\limits_{n\to\infty}\int_0^b\dfrac{\left(\sqrt[3]{x}\right)^{n+1}}{1+\sqrt[3]{x}}dx=0, \forall b\in[0,1]$.

e) Se notează $\sqrt[3]{x}=t\Rightarrow x=t^3=\varphi(t)\Rightarrow dx=3t^2dt$. Dacă x = 0 $\Rightarrow t=0$, $x=b\Rightarrow t=\sqrt[3]{b}$. Obținem

$$I=\int_0^{\sqrt[3]{b}}\frac{3t^2}{1+t}dt=3\int_0^{\sqrt[3]{b}}\frac{t^2-1+1}{t+1}dt=3\int_0^{\sqrt[3]{b}}\left(t-1+\frac{1}{t+1}\right)dt=3\cdot\left(\frac{1}{2}t^2-t+\ln(t+1)\right)\Bigg|_0^{\sqrt[3]{b}}=$$
$$=3\cdot\left(\frac{1}{2}\cdot\sqrt[3]{b}^{\,2}-\sqrt[3]{b}+\ln\left(\sqrt[3]{b}+1\right)\right).$$

f) Integrând relația de la punctul b) de la 0 la x obținem :

$$\int_0^x \frac{1}{1+\sqrt[3]{t}}dt = x+\frac{(-1)^1 x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+...+\frac{(-1)^n \cdot x^{\frac{n}{3}+1}}{\frac{n}{3}+1}+(-1)^{n+1}\int_0^x \frac{\left(\sqrt[3]{t}\right)^{n+1}}{1+\sqrt[3]{t}}dt \overset{not}{=}$$

$$= a_n(x)+(-1)^{n+1}\cdot\int_0^1 \frac{\left(\sqrt[3]{t}\right)^{n+1}}{1+\sqrt[3]{t}}dt \Leftrightarrow a_n(x)=\int_0^x \frac{1}{1+\sqrt[3]{t}}dt+(-1)^n\int_0^x \frac{\left(\sqrt[3]{t}\right)^{n+1}}{1+\sqrt[3]{t}}dt\ .$$

Din d) $\Rightarrow \lim\limits_{n\to\infty}(-1)^n\int_0^x \frac{\left(\sqrt[3]{t}\right)^{n+1}}{1+\sqrt[3]{t}}dt = 0, \forall \mathrm{x}\in[0,1]$, deci $\lim\limits_{n\to\infty} a_n(x)=\int_0^x \frac{1}{1+\sqrt[3]{t}}dt, \forall x\in[0,1]$.

g) Din punctul f) $\Rightarrow \lim\limits_{n\to\infty} a_n(x)=\int_0^x \frac{1}{1+\sqrt[3]{t}}dt, \forall x\in[0,1]$. Funcția $g(x)=\int_0^x \frac{1}{1+\sqrt[3]{t}}dt$ este o funcție strict crescătoare pe [0, 1] și continuă.. Dacă $r\in(g(0),g(1))\cap\mathbf{Q}$ atunci din proprietatea lui Darboux $\Rightarrow \exists x\in(0,1)$ astfel incat $g(x)=r$, deci

$$\lim_{n\to\infty}\left(x+\frac{(-1)^1 x^{\frac{1}{3}+1}}{\frac{1}{3}+1}+\frac{(-1)^2 x^{\frac{2}{3}+1}}{\frac{2}{3}+1}+...+\frac{(-1)^n\cdot x^{\frac{n}{3}+1}}{\frac{n}{3}+1}\right)\in\mathbf{Q}.$$